EXT I 2006 TERM 1

QUESTION ONE - (Start a new page) **marks**

(a) Differentiate $y = \ln(\cos^2 x)$ **2**

(b) The sides of a cube are decreasing at a constant rate of 2.5cms^{-1}. Find the rate at which the volume of the cube is changing when the sides are 15cm. **2**

(c) Show that $k(4k+1)^{-1} + (4k+1)^{-1}(4k+5)^{-1} = (k+1)(4k+5)^{-1}$ **2**

(d) Find the exact sum of the first twenty terms of the series: **3**

$$\log_a 4 + \log_a 16 + \log_a 64 + \ldots$$

QUESTION TWO - (Start a new page)

(a) Solve: (i) $\cos^2 x - \sin 2x = 0$ for $0 \le x \le 2\pi$ **3**

(ii) $1 = 2\log_{10} x - \log_{10}(\frac{x}{10} + 24)$ **3**

(b) Differentiate $3xe^x$ with respect to x, and hence or otherwise evaluate $\int_0^2 xe^x\,dx$. **3**

QUESTION THREE - (Start a new page)

(a) Water is pouring into a cone shaped funnel at a constant rate of $36\text{cm}^3\text{s}^{-1}$. If the diameter of the funnel is ¾ of its height, find the rate at which the depth of water is increasing when the height is 12cm. Give your answer correct to three sig. figures. **3**

(b) A dinghy is being pulled towards a wharf at a constant rate of 15m per minute. The rope is tied to the dinghy and the dinghy is 5m below the wharf. Find the rate at which the:

(i) rope is being drawn in when the dinghy is 12m from the wharf. **3**

(ii) the angle between the rope and wharf is changing when the dinghy is 12m from the wharf. **3**

QUESTION FOUR - (Start a new page)

(a) If $n! > 2^n$ for all integer values of n greater than 3, prove that $(n+1)! > 2^{n+1}$ **3**

(b) Given that $\int_0^k \frac{3x^2}{x^3+3}\,dx = \ln 10$, find the exact value of k. **2**

(c) Prove by Mathematical Induction, that $n^3 + 2n$ is divisible by 3, for all positive integers n. **4**

QUESTION FIVE - (Start a new page)

Dwayne borrows \$200 000 which is to be repaid in equal monthly repayments of $\$x$ over 20 years. If interest is charged at 6% p.a. calculated monthly on the balance outstanding, find:

(a) The amount owing after the first repayment. **1**

(b) The amount of each monthly repayment to the nearest dollar. **3**

(c) How long it would take to repay the same loan if Dwayne pays an extra \$100 every month from the very start? **3**

(d) Assuming Dwayne makes the extra repayments of \$100, and after 5 years he wins \$50 000, can he pay out the balance of the loan? If not, how much more does he owe? **2**

QUESTION SIX - (Start a new page)

(a) Express $0.\dot{5}\dot{0}$ as a geometric series and hence convert $0.\dot{5}\dot{0}$ to a rational number in its simplest form. **2**

(b) Use Simpson's Rule with 5 functional values, to find the approximate area under the curve $y = \sin(e^{2x})$, the x-axis and the lines $x = 1$ to $x = 3$. Give your answer correct to two decimal places. **4**

QUESTION SIX - continued

(c) Find the exact volume of the solid of revolution when the area under the curve $y = \cos 3x$, from $x = 0$ to $x = \frac{\pi}{6}$ is rotated about the x-axis. **3**

QUESTION SEVEN - (Start a new page)

(a) Given that the sum of the infinite geometric series $1 + 2^n + 2^{2n} + \ldots\ldots$ is 2. Find the exact value of n. **2**

(b) Find $\int_0^1 e^{\ln 4x}\,dx$. **2**

(c) MON is a quadrant of a circle centre O and radius 20cm. P is a point on the arc MN rotating about O at a constant rate, moving from M to N in 15 minutes. A is the total area of ΔOMP and ΔONP in cm^2.

(i) Show that $A = 200(\sin\theta + \cos\theta)$, where θ is the angle MOP. **1**

(ii) Find the exact rate at which A is changing when $\theta = \frac{\pi}{6}$. **4**

END OF PAPER

Yr 12 – Trm 1 2006 Solutions

Question One

(a) $y = \ln(\cos^2 x)$

$\frac{dy}{dx} = \frac{-2\cos x \sin x}{\cos^2 x}$ (1)

$= -2\tan x$ (1)

(b) $\frac{dx}{dt} = -2.5$ $\qquad V = x^3$

$\frac{dV}{dt} = ?$ $\qquad \frac{dV}{dx} = 3x^2$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$ (1)

$= 3x^2 \times -2.5$

$= -7.5x^2$

When $x = 15\text{cm}$, $\frac{dV}{dt} = -1687.5$ (1)

$\therefore$ Volume is decreasing at $1687.5\text{ cm}^3/\text{s}$

(c) $\frac{k}{(4k+1)} + \frac{1}{(4k+1)(4k+5)} = \frac{k+1}{(4k+5)}$

$\text{LHS} = \frac{k(4k+5)+1}{(4k+1)(4k+5)}$

$= \frac{4k^2+5k+1}{(4k+1)(4k+5)}$ (1)

$= \frac{(4k+1)(k+1)}{(4k+1)(4k+5)}$

$= \frac{k+1}{(4k+5)}$ (1)

$= \text{RHS}$

(d) $a = \log_a 4 = 2\log_a 2$ (1)

$2\log_a 2 + 4\log_a 2 + 6\log_a 2 + \text{-----}$

$S_{20} = \frac{20}{2}\{4\log_a 2 + 19(2\log_a 2)\}$ (1)

$= 420\log_a 2$ (1)

Question Two

(a) (i) $\cos^2 x - \sin 2x = 0$

$\cos^2 x - 2\sin x\cos x = 0$

$\cos x(\cos x - 2\sin x) = 0$ (1)

$\cos x = 0$ or $\tan x = \frac{1}{2}$ (1)

(1) $\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $x = 0.4636, 3.6052$

(ii) $1 = 2\log_{10} x - \log_{10}\left(\frac{x}{10} + 24\right)$

$1 = \log_{10} x^2 - \log_{10}\left(\frac{x}{10} + 24\right)$

$1 = \log_{10}\left(\frac{x^2}{x/10 + 24}\right)$ (1)

$10 = \frac{x^2}{\frac{x}{10} + 24}$

$x^2 = x + 240$

$0 = x^2 - x - 240$

$0 = (x-16)(x+15)$ (1)

$\therefore x = 16$ or -15

but $x > 0$ as you can't take a log of a negative number $\therefore x = 16$ only (1)

(b) $y = 3xe^x$

$\frac{dy}{dx} = 3xe^x + 3e^x$ (1)

$\int_0^2 \frac{d(3xe^x)}{dx}\,dx = \int_0^2 3xe^x\,dx + \int_0^2 3e^x\,dx$

$\frac{1}{3}\left[3xe^x\right]_0^2 = \int_0^2 3xe^x\,dx + \left[e^x\right]_0^2$ (1)

$2e^2 - 0 = \int_0^2 3xe^x\,dx + e^2 - 1$

$\therefore \int_0^2 3xe^x\,dx = e^2 + 1$ (1)

Question 3

a) $\frac{dV}{dt} = 36 cm^3/s$

$2r = \frac{3}{4}h$

$r = \frac{3}{8}h$ 1

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{\pi}{3} \cdot \frac{9h^3}{64}$

$V = \frac{3\pi h^3}{64}$

1 $\frac{dV}{dh} = \frac{9\pi h^2}{64}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{64}{9\pi h^2} \times 36$

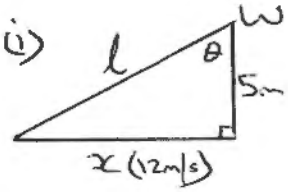
$= \frac{256}{\pi h^2}$ 1

when $h=12$, $\frac{dh}{dt} = 0.565884242$

∴ Height's increasing at 0.566 cm/s

(* This question is not the target question for rounding off errors!).

b) (i)



$l = \sqrt{25+x^2}$ (Pythagoras)

$\frac{dl}{dx} = \frac{x}{\sqrt{25+x^2}}$ 1

$\frac{dx}{dt} = 15 m/min$ $\qquad \frac{dl}{dt} = ??$

$\frac{dl}{dt} = \frac{dl}{dx} \times \frac{dx}{dt}$

$= \frac{x}{\sqrt{25+x^2}} \times -15$ 1

when $x=12$, $\frac{dl}{dt} = -13\frac{11}{13}$ 1

∴ rope is being drawn in at $13\frac{11}{13}$ m/min

ii) $\tan\theta = x/5$

$x = 5\tan\theta$ $\qquad \cos\theta = \frac{5}{13}$

$\frac{dx}{d\theta} = 5\sec^2\theta$ 1

$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$= \frac{1}{5}\cos^2\theta \times -15$ 1

1 $= \frac{1}{5} \times \frac{25}{169} \times -15$

$= \frac{-75}{169}$

∴ Angle's decreasing at $\frac{75}{169}$ rad/min.

Question 4

(a) given $n! > 2^n$ for $n \geq 4$

prove $(n+1)! > 2^{n+1}$

now $(n+1)! = (n+1)(n!)$ 1

so $(n+1)n! > (n+1)2^n$ (as $n! > 2^n$)

and $(n+1)2^n > 2(2^n)$ as $n \geq 4$ 1

so, $(n+1)n! > (n+1)2^n > 2^{n+1}$

$\therefore (n+1)n! > 2^{n+1}$ 1

ie $(n+1)! > 2^{n+1}$

b) $\int_0^K \frac{3x^2}{x^3+3}\,dx = \ln 10$

$\ln 10 = \left[\ln(x^3+3)\right]_0^K$ 1

$\ln 10 = \ln(K^3+3) - \ln 3$

$\ln 30 = \ln(K^3+3)$

$K^3 + 3 = 30$

$K^3 = 27$

$k = 3$ 1

c) step 1: Prove true for $n=1$,

$(1+2) = 3$ 1

∴ which is divisible by 3 ∴ true for $n=1$

step 2: Assume true for $n=K$, 1

ie $K^3 + 2K$ is divisible by 3

$\therefore K^3 + 2k = 3M$ (where M is an integer)

step 3: Prove true for $n=k+1$,

ie show that $(K+1)^3 + 2(K+1)$ is divisible by

LHS $= K^3 + 3K^2 + 3K + 1 + 2K + 2$ 1

$= K^3 + 2K + 3K^2 + 3K + 3$

$= 3M + 3(K^2+K+1)$ (by assumption)

$= 3(M + K^2 + K + 1)$ 1

which is divisible by 3

step 4: Since true for $n=1$, ~

Question 5

(a) let A_n = amount owing after n^{th} repayment

$A_1 = \$200000(1 + \frac{6}{1200}) - P$

$A_1 = 200000(1 + \frac{1}{200}) - x$ (1)

(b) $A_2 = 200000(1+\frac{1}{200})^2 - x[(1+\frac{1}{200}) + 1]$

⋮

$A_{240} = 200000(1\frac{1}{200})^{240} - x[(1\frac{1}{200})^{239} + \ldots + 1]$

but $A_{240} = 0$

(1) $\therefore x = \dfrac{200000(1\frac{1}{200})^{240}}{1\frac{1}{200}^{239} + 1\frac{1}{200}^{238} + \ldots + 1}$

denom. is a G.P.; $a = 1$, $r = 1\frac{1}{200}$, $n = 240$

(1) $S_{240} = \dfrac{1(1\frac{1}{200}^{240} - 1)}{\frac{1}{200}} = 462.0409$

$\therefore x = \dfrac{200000(1\frac{1}{200})^{240}}{462.0409}$

(1) $= \$1432.86$

$\therefore x = \$1433$ (nearest dollar

c) $x = 1433 + 100 = \$1533$

$n = ?$

(1) $1532 = \dfrac{200000(1\frac{1}{200})^n}{\frac{(1\frac{1}{200})^n - 1}{1/200}}$

$1.532 = \dfrac{(1\frac{1}{200})^n}{(1\frac{1}{200})^n - 1}$

$1.532(1\frac{1}{200})^n - 1.532 = (1\frac{1}{200})^n$

$0.532(1\frac{1}{200})^n = 1.532$

(1) $1\frac{1}{200}^n = 2.879699248$

$\log_{1\frac{1}{200}} 2.87969 = n$

(1) $n = \dfrac{\log_{10} 2.87969}{\log_{10} 1\frac{1}{200}}$

$n = 212.06493$

$\therefore$ It takes 212 months.

d) $A_{60} = 200000(1\frac{1}{200})^{60} - 1533(1\frac{1}{200}^{59} + \ldots + 1)$

$= 200000(1.005)^{60} - 1533\left(\dfrac{1\frac{1}{200}^{60} - 1}{\frac{1}{200}}\right)$

(1) $= \$162812.57$

$A_{60} - \$50000 = \$112812.57$

$\therefore$ Dwayne can't pay off the loan

(1) as he still owes $\$112812.57$.

Question 6

a) $0.\dot{5}\dot{0} = 0.50 + 0.0050 + 0.00005$ 1

$r = \frac{1}{100} \quad a = \frac{1}{2}$

$S_\infty = \dfrac{\frac{1}{2}}{1 - \frac{1}{100}}$

$= \dfrac{50}{99}$ 1

b) $A = \displaystyle\int_1^3 \sin(e^{2x})\,dx$

1 $A = \frac{2-1}{6}\left[f(1) + 4f(1\tfrac{1}{2}) + 2f(2) + 4f(2\tfrac{1}{2}) + f(3)\right]$

$= \frac{1}{6}\left[\sin(e^2) + 4\sin(e^3) + 2\sin(e^4) + 4\sin(e^5) + \sin(e^6)\right]$

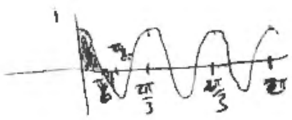
1 $= \frac{1}{6}\left[\sin(7.38906) + 4\sin(20.0855) + 2\sin(54.598) + 4\sin(148.4132) + \sin(403.4288)\right]$

$= 1.028251124$ 1

* $= 1.03$ (2 dec. places) 1

(* This is the target question if they have answered correct to 2 dec. places, deduct a mark.)

c) $V = \pi \displaystyle\int_0^{\pi/6} \cos^2 3x\,dx$



$= \frac{\pi}{2}\displaystyle\int_0^{\pi/6} (\cos 6x + 1)\,dx$ 1

$= \frac{\pi}{2}\left[\frac{1}{6}\sin 6x + x\right]_0^{\pi/6}$ 1

$= \frac{\pi}{2}\left[\left(\frac{1}{6}\sin\pi + \frac{\pi}{6}\right) - (0 + 0)\right]$

$= \frac{\pi^2}{12}$ units^3 1

QUESTION SEVEN

a) $1 + 2^n + 2^{2n} + \dots = 2$

$a=1 \quad r=2^n$

$S_\infty = \frac{a}{1-r}$

$2 = \frac{1}{1-2^n}$ (1)

$2(1-2^n)=1$

$2-2^{n+1}=1$

$2^{n+1}=1$

$\therefore n+1=0$ (1)

$\therefore n=-1$

b) $\int_0^1 e^{\ln 4x}\,dx = \int_0^1 4x\,dx$ (1)

$= \left[2x^2\right]_0^1$

$= 2-0$

$= 2$ (1)

c) i) $A = \frac{1}{2}r^2\sin\theta + \frac{1}{2}r^2\sin(90-\theta)$

$= \frac{1}{2}(20)^2\sin\theta + \frac{1}{2}(20)^2\cos\theta$ (1)

$= 200(\sin\theta + \cos\theta)$

ii) $\frac{dA}{d\theta} = 200(\cos\theta - \sin\theta)$

$\frac{dl}{dt} = \frac{1}{4}\frac{(2\pi 20)}{15}$ /min

$= \frac{10\pi}{15}$

$= \frac{2\pi}{3}$ cm/min (1)

$l = 20\theta \quad (l = r\theta)$

$\frac{dl}{d\theta} = 20$

$\frac{d\theta}{dt} = \frac{d\theta}{dl} \times \frac{dl}{dt}$

$= \frac{1}{20} \times \frac{2\pi}{3}$

$= \frac{\pi}{30}$ (1)

$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$= 200(\cos\theta - \sin\theta) \times \frac{\pi}{30}$

$= \frac{20\pi}{3}(\cos\theta - \sin\theta)$ (1)

when $\theta = \pi/6$, $\frac{dA}{dt} = \frac{20\pi}{3}\left(\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\right)$

$= \frac{20\pi}{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

$= \frac{10\pi}{3}(\sqrt{3}-1)$ (1)

$\therefore$ Area is increasing at $\frac{10\pi}{3}(\sqrt{3}-1)$ cm²/s.

End of Paper!!